Exercise 2.6

The Saint Venant equations

This document provides an outline for the solution of Exercise 2.5 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider a rectangular channel, the length and bottom slope of which are denoted by L and S_0 respectively. The elevation of the bottom at the left-hand end of the channel is denoted by z_L. The water is initially at rest. The elevation of the free surface is denoted by ζ_0 (Figure 2.25). The effect of friction is neglected and the perturbations in the free surface elevation are assumed to be small enough for the celerities to be considered independent from time. The numerical values of the physical parameters can be found in Table 2.2.

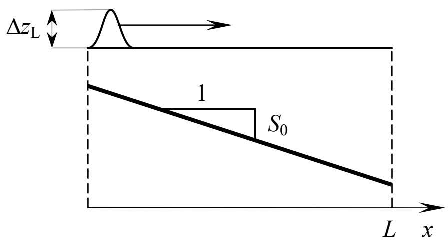

Figure 2.25. *Propagation of a perturbation in a channel with constant bottom slope.*

At $t = 0$, a perturbation appears at the left-hand end of the channel. The height of the perturbation is denoted by Δz_L.

1) Provide the expressions of the resulting perturbations Δu_L and ΔQ_L in the velocity and in the discharge.

2) Provide a graphical representation of the characteristic along which the perturbation travels in the phase space. Provide the expression for the time T_R at which the perturbation reaches the right-hand end of the channel.

3) Compute the height Δz_R of the perturbation when it reaches the right-hand end of the channel, as well as the perturbations Δu_R and ΔQ_R in the velocity and in the

discharge. What is your conclusion about the validity of the assumption that the celerity does not depend on time?

Symbol	Meaning	Value
b	Channel width	10 m
g	Gravitational acceleration	9.81 m/s^2
L	Channel length	100 m
S_0	Channel bottom slope	10 %
z_G	Elevation of the channel bottom at the left-hand end	0 m
Δz_G	Height of the perturbation at the left-hand end of the channel	0.1 m
ζ_0	Initial elevation of the free surface	1 m

Table 2.2. *Parameters for Exercise 2.6.*

2. Solution

2.1. *Question 1*

The characteristic dx/dt is used to connect the two sides of the perturbation. Both sides may be made arbitrarily close to each other, so that the integral of the source term can be assumed to be zero when integrating the characteristic relationship [2.139]

$$u_L - 2c_L = u_{L,0} - 2c_{L,0} \qquad [1]$$

where c_L et u_L denote the value of c and u immediately behind the perturbation and $c_{L,0}$ and $u_{L,0}$ denote the initial values of c and u at the left-hand boundary. Eq. [1] can be rewritten as

$$u_L - u_{L,0} = 2(c_L - c_{L,0}) \qquad [2]$$

that is

$$\Delta u_L = 2[g(\zeta_0 + \Delta\zeta - z_L)]^{1/2} - 2[g(\zeta_0 - z_L)]^{1/2} \qquad [3]$$

The discharge is given by

$$Q_L = [g(\zeta_0 + \Delta\zeta - z_L)]^{1/2} u_L \qquad [4]$$

The data available in Table 2.2 lead to Δu_L = 0,31 m/s and ΔQ_L = 3,36 m/s.

2.2. *Question 2*

The assumption that the wave celrity is not modified significantly by the discharge perturbation means that the wave celerities remain equal to their initial values at all points. Since the initial flow velocity is zero, the equation of the characteristic that travels to the right simplifies into

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda^{(2)} \approx c_0(x, t = 0) = [(\xi_0 - z_b(x))g]^{1/2} \qquad [5]$$

where c_0 denotes the value orf c at $t = 0$. Note that c_0 is a function of x. The bottom slope S_0 being constant, Eq. [5] becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = [(\xi_0 - z_\mathrm{L} + S_0 x)g]^{1/2} \qquad [6]$$

Integrating Eq. [6] leads to

$$t - t_0 = 2\left(\frac{x}{gS_0} + \frac{\zeta_0 - h_\mathrm{L}}{gS_0^2}\right)^{1/2} - 2\left(\frac{x_0}{gS_0} + \frac{\zeta_0 - h_\mathrm{L}}{gS_0^2}\right)^{1/2} \qquad [7]$$

This is the equation of the characteristic that passes at (x_0, t_0) in the phase space. In the case where $x_0 = 0$ and $t_0 = 0$, Eq. [7] simplifies into

$$t = 2\left(\frac{x}{gS_0} + \frac{\zeta_0 - h_\mathrm{L}}{gS_0^2}\right)^{1/2} - 2\left(\frac{\zeta_0 - h_\mathrm{L}}{gS_0^2}\right)^{1/2} \qquad [8]$$

The characteristic is illustrated in the phase space in Figure 1. From the data available in Table 2.2, the wave reaches the right-hand end of the channel at $t = 14.8$ sec.

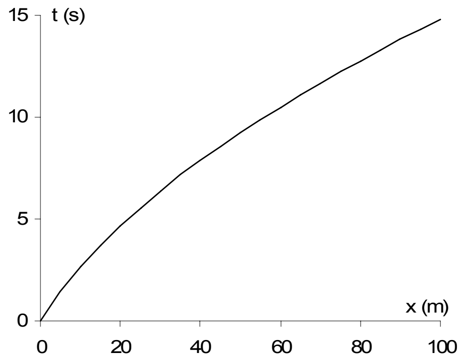

Figure 1. *Representation of the trajectory of the perturbation in the phase space.*

2.3. *Question 3*

Assuming that friction is negligible, the following relationship holds along the characteristic $\mathrm{d}x/\mathrm{d}t = c_0(x)$ passing at $(x = 0, t = 0)$ (Figure 2)

$$\frac{\mathrm{d}}{\mathrm{d}t}(u + 2c) = S_0 g \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c_0 \qquad [9]$$

where u and c denote the flow velocity and the speed of the waves in still water immediately behind the perturbation. Since the primary objective is to find out how the perturbation varies in space, Eq. [9] is transformed into a differential equation with respect to x by noticing that $\mathrm{d}x = c_0\,\mathrm{d}t$, hence $\mathrm{d}/\mathrm{d}t = c_0\,\mathrm{d}/\mathrm{d}x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(u + 2c) = \frac{gS_0}{c_0} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c_0 \qquad [10]$$

Moreover, each point immediately ahead of the characteristic dx/dt = + c is connected to a point immediately behind the characteristic by a characteristic dx/dt = – c (dashed line in Figure 2).

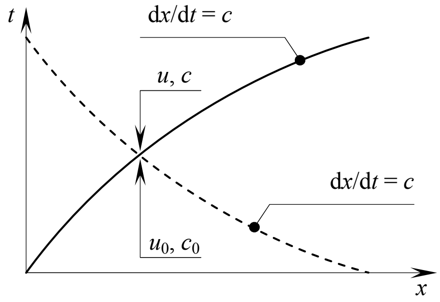

Figure 2. *Positive and negative characteristics in the phase space.*

Integrating Eq. [2.139] over the zero distance that separates both sides of the perturbation yields

$$u - 2c = u_0 - 2c_0 \qquad [11]$$

Rememembering from Eq. [5] that

$$c_0(x) = [(\zeta_0 - z_L + S_0 x)g]^{1/2} \qquad [12]$$

differentiating [11] along the characteristic dx/dt = c_0 gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(u - 2c) = -2\frac{\mathrm{d}c_0}{\mathrm{d}x} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \qquad [13]$$

Combining Eqs. [10] and [14] yields the following system

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{S_0 g}{2c} - \frac{\mathrm{d}c_0}{\mathrm{d}x} && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ \frac{\mathrm{d}c}{\mathrm{d}x} &= \frac{S_0 g}{4c} + \frac{1}{2}\frac{\mathrm{d}c_0}{\mathrm{d}x} && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \end{aligned}\right\} \qquad [14]$$

From Eq. [12]

$$\frac{\mathrm{d}c_0}{\mathrm{d}x} = \frac{1}{2}\frac{g\dfrac{\mathrm{d}h_0}{\mathrm{d}x}}{[(\zeta_0 - z_b(x))g]^{1/2}} = \frac{gS_0}{2c_0} \qquad [15]$$

Eq. [15] holds because the initial free surface is horizontal. The assumption of a constant bottom slope is not needed. Substituting Eqs. [12] and [15] into [14]leads to

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x} &= 0 && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ \frac{\mathrm{d}c}{\mathrm{d}x} &= \frac{\mathrm{d}c_0}{\mathrm{d}x} && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \end{aligned}\right\} \qquad [16]$$

In other words

$$\left.\begin{aligned} u(x) &= u_{\mathrm{L}} && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ c(x) - c_0(x) &= c_{\mathrm{L}} - c_0(0) && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \end{aligned}\right\} \qquad [17]$$

The perturbations in both u and c remain unchanged as they travel to the left. Since $\mathrm{d}h = 2c/g\ \mathrm{d}c$, the perturbation in the water depth increases as the water depth increases (see the spreadsheet).

You may find such a result disturbing, in which case you would be right. Indeed, a wave travelling inot deeper water disturbs a water column with an increased inertia. Consequently, the amplitude of the perturbation should be expected to decrease. The reason for the apparently unphysical behaviour is to be found in the assumptions adopted for this exercise. One of these assumptions is that the wave celerity is identical to c, which is assumed to be constant in time. This is wrong because the celerity, that is equal to $u + c$ (or $u - c$ if the negative wave is considered), changes when the perturbation. Neglecting the variations in the celerity leads to unphysical results.

This conclusion is confirmed by Figure 3, that shows the simulated propagation of a disturbance of height 1 cm in a canal with a bottom slope of 1%. The simulation is carried out using the first-order Method Of Characteristics (MOC) presented in Chapter 6 of the book (see application exercise 6.2).

(a) (b)

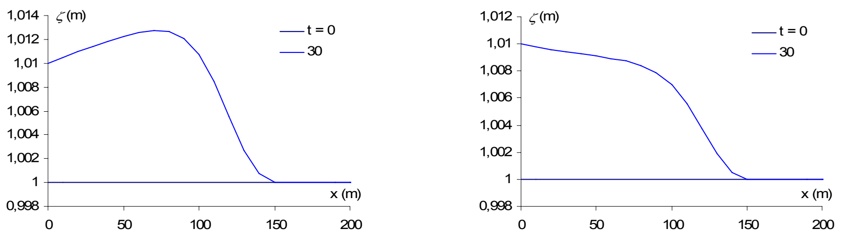

Figure 3. *Water level computed using the MOC. Wave celerity computed from the initial condition (a), updated at each time step (b).*

The profile obtained assuming that the celerity is constant in time is displayed in Figure 3. As expected from Eq. [17], the height of the perturbation increases as the wave travels downstream, which is unphysical. In contrast, recomputing the value of the celerity from the local value of u and c at all times yields a much more physical solution, as shown in Figure 3b.

This simple example shows that simplifying assumptions must be checked carefully, otherwise leading to unphysical results.

N.B. : such a problem does not occur in the case of the water hammer equations (see Exercise 2.3) because in the case of the water hammer the celerity does not change with time.