Exercise 2.2

Water hammer

This document provides an outline for the solution of Exercise 2.2 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider a horizontal pipe of cross-sectional area A, where the celerity of the pressure waves is piecewise constant. The celerity to the left of the point $x = x_0$ is denoted by c_1, the celerity to the right of $x = x_0$ is denoted by c_2. The fluid is initially at rest, the pressure is uniformly equal to p_0. The influence of friction is assumed to be negligible.

At $t = 0$ the pressure at the left-hand end of the pipe rises instantaneously to the constant value p_1. The resulting pressure discontinuity propagates to the right at a speed c_1.

1) Derive the expression of the discharge Q_1 on the left-hand side of the pressure discontinuity.

2) The pressure wave reaches the abscissa x_0 where the sound speed changes to c_2. Considering that the pressure is continuous at $x = x_0$, show that the pressure and the discharge change to new values p_2 and Q_2 when the pressure wave reaches $x = x_0$. Provide the expression of p_2 and Q_2 as functions of p_0, p_1, c_1 and c_2.

3) Show that the pressure surge is amplified if $c_1 < c_2$ (in other words, $|p_2 - p_0| > |p_1 - p_0|$). Conversely, show that the pressure surge is damped if $c_1 > c_2$. Provide a physical interpretation for such a behaviour.

2. Solution

2.1. *Question 1*

Equations [2.82] are used along the first characteristic between the regions (p_0, u_0) and (p_1, u_1) (Figure 1).

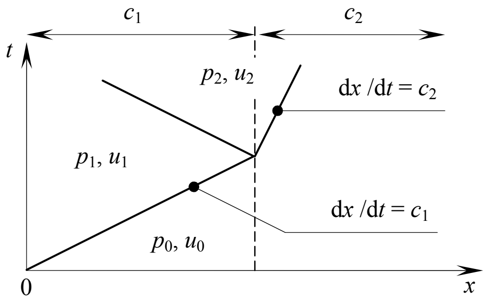

Figure 1. *Pressure wave. Representation in the phase space.*

$$p_1 - p_0 = (u_1 - u_0)\rho c_1 \quad [1]$$

Multiplying Eq. [1] by the pipe cross-sectional area, noticing that $Q_0 = 0$ leads to

$$Q_1 = \frac{A}{\rho c_1}(p_1 - p_0) \quad [2]$$

2.2. ***Question 2***

That the pressure is continuous may be justified as follows. Consider an elementary control volume that contains the point where the celerity changes. Under steady state, the volume is subjected to a zero acceleration (by definition of steady state). From the fundamental principle of dynamics, the pressure is identical on both sides of the control volume. Since the control volume can be made arbitrarily small, the pressure is continuous at the celerity discontinuity.

The regions (p_1, u_1) and (p_2, u_2) are connected by a characteristic $dx/dt = c_1$. Conversely, the regions (p_0, u_0) and (p_2, u_2) are connected by a characteristic $dx/dt = - c_2$. This leads to the following system

$$\left.\begin{aligned} p_2 + \rho c_1 u_2 &= p_1 + \rho c_1 u_1 \\ p_2 - \rho c_2 u_2 &= p_0 - \rho c_2 u_0 \end{aligned}\right\} \quad [3]$$

Solving the system [3] for p_2 yields

$$p_2 = \frac{c_2 p_1 + c_1 p_0}{c_1 + c_1} + \frac{\rho c_1 c_2}{c_1 + c_1}(u_1 - u_0) \quad [4]$$

Substituting [1] into [4] leads to the following relationship

$$p_2 = \frac{2c_2}{c_1 + c_1} p_1 + \frac{2c_1}{c_1 + c_1} p_0 \quad [5]$$

2.3. ***Question 3***

Eq. [5] may be rewritten as

$$p_2 - p_0 = \frac{2c_2}{c_1 + c_1}(p_1 - p_0) \qquad [6]$$

If $c_2 > c_1$ the pressure variation is amplified. If $c_2 < c_1$ the pressure variation is damped. The following explanation may be given for this behaviour.

Consider a pressure variation travelling from left to right (Figure 2a). If c_2 is larger than c_1, the right-hand part of the pipe is more rigid than the left-hand part. Therefore the pipe is less deformable in the right-hand part than in the left-hand part. When the pressure wave arrives at the point where c changes from c_1 to c_2 (Figure 2b) the pipe $c = c_2$ is not deformable enough to accommodate the discharge Q_1 and the transmitted discharge Q_2 is smaller than Q_1. The accumulating water immediately on the left-hand side of the discontinuity triggers the inflation of the pipe (Figure 2c), with a reflected pressure wave propagating to the left.

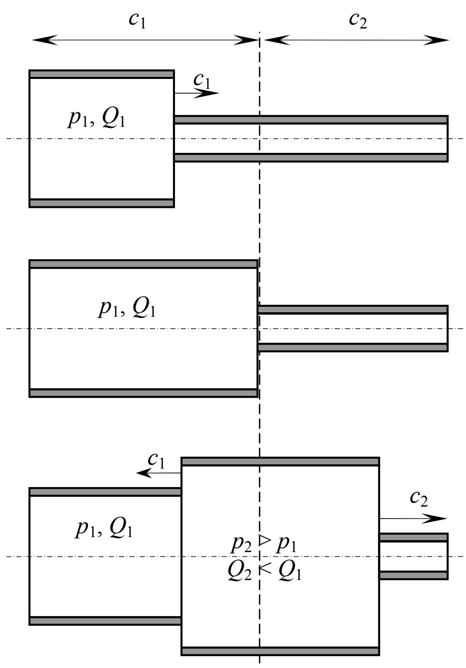

Figure 2. *Surpression parvenant dans une région de célérité supérieure.*

Note that Figure 2 (bottom) is slightly inaccurate. Indeed, since $c_1 < c_2$, the pipe is slightly more inflated in the region $c = c_1$ than in the region $c = c_2$, that is less deformable.