

Exercise 1.4

The Buckley-Leverett equation

This document provides an outline for the solution of Exercise 1.4 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider an aquifer of length L contaminated by a hydrocarbon at an initial water saturation of 50 % (Figure 1.27). The remediation technique consists in injecting pure water at the left-hand boundary of the aquifer ($x = 0$). The Darcy velocity V is constant.

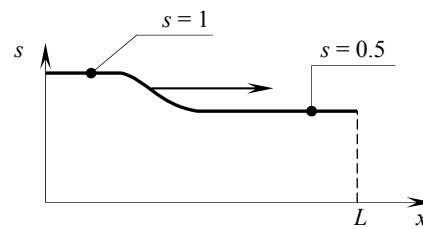


Figure 1.27. Injecting pure water to decontaminate an aquifer.

1) Determine the shape and the expression of the saturation profile as a function of space (hint: try to express x as a function of s as opposed to the usual formulation $s(x)$).

2) Determine the time after which the average hydrocarbon saturation is 5 %, 1 % and 0.5 %. Carry out the numerical application for the parameter values in Table 1.5.

Symbol	Meaning	Value
b_{BL}	Shape parameter for the Buckley-Leverett flux function	1
L	Length of the aquifer	200 m
s_0	Initial water saturation in the aquifer	0.5
s_L	Water saturation at the left-hand boundary ($x = 0$)	1
V	Darcy velocity	1 m/day

Table 1.5. Parameters for Exercise 1.4.

2. Solution

2.1. Question 1

At times $t > 0$, the water saturation at $x = 0$ rises suddenly from $s_0 = 0.5$ to $s_i = 1$. The saturation profile becomes discontinuous at the left-hand boundary. The discontinuous profile may be approximated with a continuous profile where the saturation decreases from s_i to s_0 within an infinitely small distance ε (Figure 1). The profile remains continuous, so do the derivatives of s and consequently the wave celerity. The nonconservation and characteristic form of the Buckley-Leverett equation remain applicable.

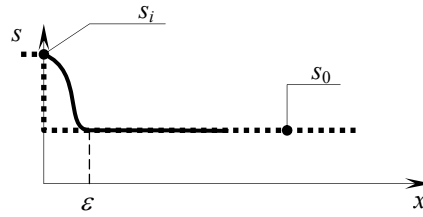


Figure 1. Approximating a discontinuous profile (dashed line) with a continuous profile (solid line).

The profile at $t = 0^+$ is given by

$$\left. \begin{array}{ll} x(t=0) = 0 & \text{for } s = s_i \\ x(t=0) \in]0, \varepsilon[& \text{for } s_0 < s < s_i \\ s(t=0) = s_0 & \text{for } x > \varepsilon \end{array} \right\} \quad [1]$$

where ε may be taken as small as deemed appropriate. In Eq. [1] x is expressed as a function of s . This unusual writing for the saturation profile makes the solution easier to compute, as shown hereafter.

At the time t , a given point in the profile with saturation s has moved by a distance $D(s, t)$ (Figure 2) given by

$$D(s, t) = \lambda(s) t \quad [2]$$

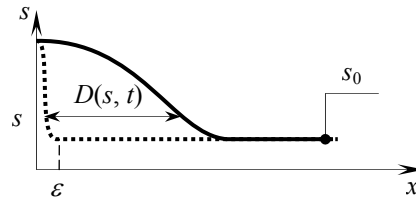


Figure 2. Saturation profile as a function of time. Initial profile (dashed line), final profile (solid line).

Note that Eq. [2] holds only because λ is a function of s only. Therefore, λ is constant along the characteristic lines because s is also constant. The profile at the time t is given by

$$\left. \begin{aligned} x(t > 0) &= \lambda(s_i)t && \text{for } s = s_i \\ x(t > 0) &\in]\lambda(s)t, \lambda(s)t + \varepsilon[&& \text{for } s_0 < s < s_i \\ s(t > 0) &= s_0 && \text{for } x > \lambda(s_0)t + \varepsilon \end{aligned} \right\} \quad [3]$$

When ε tends towards zero, one has

$$\left. \begin{aligned} x(t > 0) &= \lambda(s)t && \text{for } s_0 < s \leq s_i \\ s(t > 0) &= s_0 && \text{for } x > \lambda(s_0)t \end{aligned} \right\} \quad [4]$$

Substituting the expression of λ as given by Eq. [1.117] into [4] leads to

$$\left. \begin{aligned} x(t > 0) &= \frac{2(1-s)s}{[s^2 + (1-s)^2 b_{BL}]^2} b_{BL} V t && \text{for } s_0 < s \leq s_i \\ s(t > 0) &= s_0 && \text{for } x > \lambda(s_0)t \end{aligned} \right\} \quad [5]$$

A graphical illustration of the profile and its calculation procedure may be found in the spreadsheet <http://vincentguinot.free.fr/waves/ex14.xls>.

2.2. Question 2

it is first noticed that the decontamination front moves at the speed $\lambda(s_0)$. The time T_L at which the front reaches the downstream end of the aquifer is given by

$$T_L = \frac{L}{\lambda(s_0)} \quad [6]$$

Consequently, the concentration at the downstream end of the aquifer at times $t \leq T_L$, remains equal to the initial value s_0 , and the flux is constant, equal to $F(s_0)$. The flux at the upstream boundary being $F(s_i)$, the average water saturation \bar{s} is given by

$$\frac{d\bar{s}}{dt} = \frac{F(s_i) - F(s_0)}{L} \quad \text{for } t \leq T_L \quad [7]$$

where F is given as in Eq. [1.111] :

$$F(s) = \frac{s^2}{s^2 + (1-s)^2 b_{BL}} V$$

Therefore

$$\bar{s} = s_0 + \frac{F(s_i) - F(s_0)}{L} t \quad \text{for } t \leq T_L \quad [8]$$

At $t = T_L$, the average saturation in the aquifer is

$$\bar{s}(T_L) = s_0 + \frac{F(s_i) - F(s_0)}{\lambda(s_0)} \quad [9]$$

The numerical application gives $T_L = 100$ days and $\bar{s}(T_L) = 0.75$. When the decontamination front reaches the downstream end of the aquifer, 50% of the initial hydrocarbon quantity is still present in the aquifer.

Calculating \bar{s} for times $t \geq T_L$ is a slightly more complicated task because s is not constant at the downstream boundary of the aquifer. It thus becomes necessary to determine the saturation profile all over the aquifer, compute its integral and find the value of t for which the average concentration is equal to 95%, 99% and 99.5%. The integral of the saturation profile is best computed by plotting λ as a function of s (Figure 3).

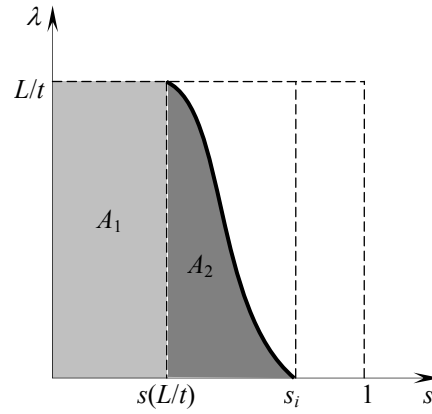


Figure 3. Computing the integral of the saturation profile by breaking the integral into two subdomains.

The average value of the saturation profile is to be computed between $\lambda = 0$ (that corresponds to $x = 0$, that is, the upstream boundary of the domain) and $\lambda = L/t$ (that corresponds to the downstream boundary of the domain, $x = L$). The area is divided into two parts A_1 and A_2 , as shown in Fig. 3

$$\left. \begin{aligned} A_1 &= \frac{L}{t} s(L/t) \\ A_2 &= \int_{s(L/t)}^{s_i} \lambda(s) ds \end{aligned} \right\} \quad [10]$$

where $s(L/t)$ denotes the value of s for which $\lambda = L/t$. The expression of l being rather complex, $s(L/t)$ is not sought analytically, only its numerical value is sought. Note that, by definition of λ , the second equation [10] becomes

$$A_2 = F(s_i) - F(s(L/t)) \quad [11]$$

The total area of the profile is

$$A = \frac{L}{t} s(L/t) + F(s_i) - F(s(L/t)) \quad [12]$$

and the average saturation is given by

$$\bar{s} = \frac{A}{L/t} = s(L/t) + \frac{F(s_i) - F(s(L/t))}{L} t \quad [13]$$

A graphical illustration of the profile and its calculation procedure may be found in the spreadsheet <http://vincentguinot.free.fr/waves/ex14.xls>.