Exercise 1.1

The inviscid Burgers equation

This documents provides an outline of the solution of Exercise 1.1 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

The Burgers equation is to be solved for the following initial conditions.

– The flow velocity is uniformly equal to u_1 for $x < x_1$. It is uniformly equal to u_2 for $x > x_2 > x_1$. It increases linearly from $u_1 < u_2$ to u_2 between x_1 and x_2.

– The density is uniformly equal to ρ_0 over the domain.

1) Derive the analytical formula for the profiles $u(x)$ and $\rho(x)$ at $t > 0$.

2) Sketch the behaviour of the solution in the physical space and in the phase space.

2. Solution outline

2.1. *Question 1*

2.1.1. *Velocity profile*

Readoning as in Subsection 1.4.3, an equation is derived for $\partial u / \partial x$ along the characteristics issued from a value of x between x_1 and x_2

$$\frac{\partial u}{\partial x}(t) = \frac{u_0^{'}}{1 + u_0^{'} t} \quad for \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [1]$$

where $u_0^{'} = (u_2 - u_1)/(x_2 - x_1)$ is the derivative of de u between x_1 and x_2 at $t = 0$. For a given time t the value of $\partial u / \partial x$ does not depend on x, which means that the velocity profile is piecewise linear.

2.1.2 *Density profile*

The densitiy profile is deduced from the continuity equation [1.62]. Expanding the derivative $\partial(\rho u) / \partial x$ in Eq. [1.62], one obtains the following equation in ρ :

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\partial u}{\partial x}\rho \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [2]$$

– Note that $\partial u / \partial x = 0$ along the characteristics issued from $x < x_1$ and $x > x_2$. Consequently, along such lines, ρ is constant, identical to the initial values.

– Along the lines issued from $x_1 \le x \le x_2$, the derivative $\partial u / \partial x$ satisfies Eq. [1]. Substituting [1] into [2] leads to

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{u_0'}{1 + u_0' t}\rho \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [3]$$

Eq. [3] is integrated into

$$\rho = \frac{1}{1 + u_0' t}\rho_0 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [4]$$

In the region that is contained between the characteristics issued from x_1 and x_2, the density is constant in space and decreases with time. This behaviour can be interpreted as follows. Since the density moves at the velocity u, the total mass between the characteristics issued from x_1 and x_2 is constant. Since the characteristic issued from x_2 moves faster than that issued from x_1, the distance between the two characteristics increases with time and the density decreases.

2.2. Question 2

Figure 1 illustrates the behaviour of the density and the velocity profile.

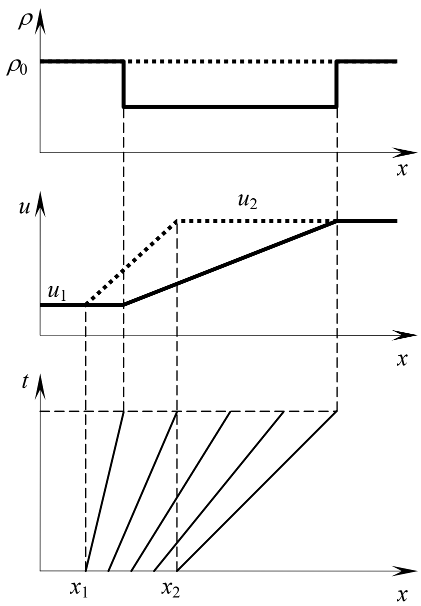

Figure 1. *Solution in the physical space (top), in the phase space (bottom).*