Exercise 6.1 (2)

Finite difference methods for scalar laws

This document provides an outline for the solution of Exercise 6.1 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE. The application to Exercise 1.2 is detailed hereafter.

1. Problem

Check the conclusions of Exercises 1.1 to 1.5 using finite difference methods. The following methods are advised:

– a characteristic-based method,

– an upwind scheme (conservative version),

– Preissmann's scheme,

– a TVD scheme.

2. Solution

2.1. *Discretization*

Only the conservation form of the upwind scheme is implemented because it eliminates the need for the calculation of the wave celerity, as well as the integral of the source term that is usually present in the characteristic form. The principle of the implementation of the other possible methods is detailed in Exercise 6.1(1).

The conservative, explicit upwind scheme as applied to the kinematic wave can be written as

$$A_i^{n+1} = A_i^n + \frac{\Delta t}{\Delta x_{i-1/2}} (Q_{i-1}^n - Q_i^n) \qquad [1]$$

where Q is computed directly from A via Strickler's law

$$Q_i^n = K_{i-1/2} S_{0,i-1/2}^{1/2} (R_H^{2/3} A)_i^n \qquad [2]$$

and the cell size $\Delta x_{i-1/2}$ is given by

$$\Delta x_{i-1/2} = \frac{x_{i+1} - x_{i-1}}{2} \qquad [3]$$

The hydraulic radius R_H is defined as

$$R_H = \frac{(b_0 + h)h}{b_0 + 2h / \cos\theta} \qquad [4]$$

while h and A are related by the following formula

$$h = \left\{ -\frac{b_0}{2} + \left[\left(\frac{b_0}{2} \right)^2 + 4A \right]^{1/2} \right\} \frac{1}{\operatorname{tg}\theta} \qquad [5]$$

N.B. : Eq. [4] is obtained by solving the equation $(b_0 + h\ \operatorname{tg}\theta)h = A$ and retaining the positive root, which is the only physically permissible root.

The slope $S_{0,i-1/2}$ is computed as the average value between the current point and the point located immediately upstream

$$S_{0,i-1/2} = \frac{z_{b,i-1} - z_{b,i}}{\Delta x_{i-1/2}} \qquad [6]$$

2.2. *Application*

The discretization is applied to a channel with the following charcteristics (see spreadsheet http://vincentguinot.free.fr/waves/ex612.xls) :

– bottom width $b_0 = 10$ m ;

– Strickler coefficient $K_{\mathrm{Str}} = 40$;

– angle between the embankments and the vertical : $\theta = 45°$;

– bottom slope : 10% for $x < 400$ m et $x > 800$ m, 5% for 400 m $< x <$ 800 m ;

– Initial discharge $Q_0 = 125.58$ m^3 s^{-1}.

The celerity is 7.49 m s^{-1} in the upstream and downstream parts of the channel and 13.74 m s^{-1} in the middle part. The discharge at the upstream boundary is raised progressively to 130 m^3 s^{-1}, kept constant for a while, and brought back to the initial value. Figure 1 shows the profiles of h and Q at three different times. It can be seen that the discharge perturbation remains constant, even in the part of the channel where the slope changes, while the water depth changes. The difference between the profiles at $t = 34$ s and $t = 110.5$ s is a consequence of numerical diffusion.

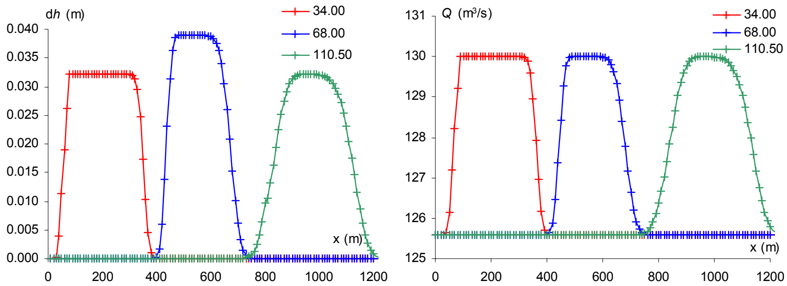

Figure 1. *Profiles of A (left) and Q (right) at three different times.*