Exercise 4.2

The Euler equations

This document provides an outline for the solution of Exercise 4.2 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider the Riemann problem formed by the Euler equations with the following initial conditions

$$\left.\begin{aligned} p_{\mathrm{L}} &= p_{\mathrm{R}} \\ \rho_{\mathrm{L}} &= \rho_{\mathrm{R}} \\ u_{\mathrm{L}} &= -u_{\mathrm{R}} \end{aligned}\right\} \qquad [4.64]$$

Show that the nature of the waves in the solution depends on the sign of u_{L}. Provide a complete description of the solution in each case (nature of the waves, expressions for the pressure, velocity and density in each of the regions). Show that a void zone may appear if u_{L} is negative, smaller than a given threshold value to be determined. Explain how the symmetry of this Riemann problem may be used to solve boundary value problems, such as the sudden closure of a valve in a pipe or the impact of a gas jet onto a wall.

2. Solution

It is first recalled that the solution of the Riemann problem for the Euler equations is made of an intermediate region where both the pressure and the flow velocity are constant. This intermediate region is separated from the left and right states by waves that may be rarefaction or shock waves. The region is made of two subregions, separated by a contact discontinuity, across which the entropy is discontinous.

2.1. *Symmetry considerations*

Symmetry considerations leads to the following remarks.

– Changing the direction of the *x*-axis leaves the Riemann problem unchanged. Consequently, the flow velocity in the intermediate region of constant state verifies $u^* = -u^*$, hence

$$u^* = 0 \quad [1]$$

– The leftward and rightward wave are identical and propagate at the same speed (only the sign of the speed is different). Determining the features of e.g. the leftward wave leads to the complete determination of the solution.

– Consequently, the contact discontinuty moves at a zero speed. The problem being symmetrical, the entropy jump is zero across the contact discontinuity.

2.2. *Case* $u_L > 0$

In such a case, both the heuristic approach and the Riemann invariants lead to the conclusion that $u^* - c^*$ in the intermediate region of constant state is larger than $u_L - c_L$ in the left state. The leftward wave should then be a shock wave. Then u^* and c^* are related to u_L and c_L via the Rankine-Hugoniot relationships

$$\left.\begin{aligned} \rho_L u_L &= (\rho_L - \rho^*) c_s \\ \rho_L u_L^2 + (p_L - p^*) &= \rho_L u_L c_s \end{aligned}\right\} \quad [2]$$

where c_s is the speed of the shock. Note that c_s is negative.

Substituting the first equation [2] into the second equation allows c_s to be eliminated. Multiplying by $\rho_L - \rho^*$ yields

$$\rho_L u_L^2 (\rho_L - \rho^*) + (p_L - p^*)(\rho_L - \rho^*) = (\rho_L u_L)^2 \quad [3]$$

The function $f(\rho)$ is defined as

$$f(\rho) = (\rho_L u_L^2 + p_L - p)(\rho_L - \rho) \quad [4]$$

ρ^* is the value of ρ such that

$$f(\rho^*) = (\rho_L u_L)^2 \quad [5]$$

Since s is a constant along the characteristic $dx/dt = u + c$ the quantity p/ρ^γ is also a constant

$$p = p_L \left(\frac{\rho}{\rho_L}\right)^\gamma \quad [6]$$

Then f can be rewritten as

$$f(\rho) = \left[\rho_L u_L^2 + p_L - p_L \left(\frac{\rho}{\rho_L}\right)^\gamma\right](\rho_L - \rho) \quad [7]$$

It is easy to check that

– $f = 0$ for $\rho = \rho_L$,

– $f = 0$ for $p = p_1$ (and $\rho = \rho_1$) such that

$$p_1 = \rho_L u_L^2 + p_L \quad [8]$$

that is

$$\rho_1^{\gamma} = \frac{\rho_L u_L^2}{p_L} + \rho_L^{\gamma} \qquad [9]$$

It is easy to check that $f < 0$ for $\rho_L < \rho < \rho_1$ and $f > 0$ for $\rho > \rho_1$. It is visible from the first equation [2] that, c_s being negative, ρ^* is larger than ρ_L. The solution is to be determined using iterative methods such as the Newton-Raphson approach. It is advised that the iterations be intialised using the Riemann invariant along $\mathrm{d}x/\mathrm{d}t = u + c$:

$$\beta_1 p^{*\beta_2} = u_L + \beta_1 p_L^{\beta_2} \qquad [10]$$

where β_1 et β_2 are given as in Eq. [2.225]. The density ρ^* is obtained from Eq. [6].

2.3. *Case $u_L < 0$*

In this case the leftward wave is a rarefaction wave. The Riemann invariants are applicable and [10] is valid. Note that a zero pressure (that is, a void region) is obtained for a limit value of u_L given by

$$u_L^{\mathrm{void}} = -\beta_1 p_L^{\beta_2} \qquad [11]$$

Any value of u_L smaller than this limit value leads to the creationof a void region, the tail of which moves at $u - c$, with $c = p = 0$ and u verifies

$$u = u_L + \beta_1 p_L^{\beta_2} \qquad [12]$$

2.4. *Exploitation des propriétés de symétrie*

As mentioned in 2.1, any situation where the flow velocitye is zero at a given point may be obtained from a Riemann problem where the left and right states are symmetrical. This property is sometimes used in finite volume methods when no-flow boundary conditions must be prescribed.