Exercise 3.5

The Euler equations

This document provides an outline for the solution of Exercise 3.5 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

An airplane moves at Mach 1 in immobile air. For the sake of simplicity, the coordinate system is attached to the airplane.

1) Write the continuity equation and the momentum equation under the assumption of steady state. The flow velocity is assumed to be zero on the hull of the airplane. Show that the assumption of a steady state flow necessarily induces a multidimensional flow pattern and that the air must be 'evacuated' in the lateral direction.

2) Determine the lateral flow, the pressure rise and the air density next to the hull. Carry out the numerical application for the parameters in Table 1.

3) Check that the entropy principle is verified across the shock.

Symbol	Meaning	Value
M_0	Far field Mach number upstream of the airplane	1
p_0	Far field pressure upstream of the airplane	10^5 Pa
γ	Polytropic constant for a perfect gas	1.4
ρ_0	Far field air density upstream of the airplane	1.2 kg/m^3

Table 3.1. *Parameters for Exercise 3.5.*

2. Solution

2.1. *Question 1*

In the coordinate system that moves with the plane, the Euler equations simplify into (note : the velocity immediately upstream fo the plane is zero)

$$\left.\begin{aligned} \rho_0 u_0 &= (\rho_0 - \rho_1)c_s \\ \rho_0 u_0^2 + p_0 - p_1 &= \rho_0 u_0 c_s \\ (E_0 + p_0)u_0 &= (E_0 - E_1)c_s \end{aligned}\right\} \quad [1]$$

where the subscript 0 denotes the far field value and the subscript 1 denotes the values immediately upstream of the airplane.

By definition, a stationary shock moves at the speed of the airplane, thus $c_s = 0$. From the first equation [1], this appears to be possible only if a lateral outflow is allowed in order to 'remove' a fixed mass per unit time from the system.

2.2. ***Question 2***

The modified equations may be written as (note that $c_s = 0$)

$$\left.\begin{aligned} \rho_0 u_0 &= q\rho_1 \\ \rho_0 u_0^2 + p_0 - p_1 &= 0 \\ (E_0 + p_0)u_0 &= qE_1 \end{aligned}\right\} \quad [2]$$

where q is the lateral outflow rate. Note that q is also to taken into account in the momentum equation via a momentum sink term $q\rho u_1$. However, since $u_1 = 0$, the momentum sink term is zero. The pressure p_1 is obtained directly from the second equation [2]

$$p_1 = p_0 + \rho_0 u_0^2 \quad [3]$$

The speed u_0 verifies $u_0 = c_0$ because the Mach number is unity. From the formula [2.221] for the speed of sound one has $\rho_0 u_0^2 = \rho_0 c_0^2 = \gamma p_0$. Eq. [3] then becomes

$$p_1 = (\gamma + 1)p_0 \quad [4]$$

The first and third equations [2] yield

$$\left.\begin{aligned} \rho_0 u_0 &= q\rho_1 \\ \left(\frac{\gamma}{2} + \frac{1}{\gamma - 1}\right)u_0 p_0 &= q\frac{p_1}{(\gamma - 1)} \end{aligned}\right\} \quad [5]$$

Substituting Eq. [4] into the second equation [5] leads to

$$q = \frac{\gamma^2 - \gamma + 2}{2(\gamma + 1)}u_0 \quad [6]$$

The density ρ_1 is obtained from [6] and the first equation [5]

$$\rho_1 = \frac{\rho_0 u_0}{q} = \rho_0 \frac{2(\gamma + 1)}{\gamma^2 - \gamma + 2} \quad [7]$$

The numerical values in the Table yield $p_1 = 2.4\times10^5$ Pa, $q = 182$ m s^{-1} and $\rho_1 = 2.25$ kg m^{-3}.

2.3. *Question 3*

The entropy s is defined as in Eq. [2.190] :

$$\mathrm{d}s = \frac{R}{\gamma - 1}\mathrm{d}\left[\ln\left(\frac{p}{\rho^\gamma}\right)\right]$$

Therefore

$$s_1 = s_0 + \frac{R}{\gamma - 1}\ln\left(\frac{p_1}{\rho_1^\gamma}\frac{\rho_0^\gamma}{p_0}\right) \qquad [8]$$

The entropy per unit volume is then

$$\rho_1 s_1 = \rho_1 s_0 + \rho_1\frac{R}{\gamma - 1}\ln\left(\frac{p_1}{\rho_1^\gamma}\frac{\rho_0^\gamma}{p_0}\right) \qquad [9]$$

and

$$\rho_1 s_1 - \rho_0 s_0 = (\rho_1 - \rho_0)s_0 + \rho_1\frac{R}{\gamma - 1}\ln\left(\frac{p_1}{\rho_1^\gamma}\frac{\rho_0^\gamma}{p_0}\right) \qquad [10]$$

The second term of the right-hand side member is equal to 4.3. Since ρ_1 is larger than ρ_0, s_0 being positive, $\rho_1 s_1$ is larger than $\rho_0 s_0$.

N.B. : This can also be shown by proving that p/ρ^γ increases across the shock.