Exercise 3.3

The Buckley-Leverett equation

This document provides an outline for the solution of Exercise 3.3 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider an aquifer, the characteristics of which are given in Table 1.5 in Exercise 1.4 (see Subsection 1.8.2.4). The aquifer is now assumed to be uniformly contaminated with an initial hydrocarbon saturation of 90 % (i.e. the initial water saturation is assumed to be 10 % everywhere). As in Exercise 1.4, the aquifer is decontaminated by injecting pure water with a Darcy velocity V at the left-hand end of the domain.

1) Show that the saturation profile at $t > 0$ is a compound wave.

2) Compute the propagation speed of the shock.

3) Compute the time at which the average contamination (i.e. the average hydrocarbon saturation) in the aquifer is 5 %, 1 % and 0.5 %.

Symbol	Meaning	Value
b_{BL}	Shape parameter in the Buckley-Leverett flux	1
L	Aquifer length	200 m
s_0	Initial water saturation	0.1
s_i	Injected water saturation	1
V	Darcy velocity	1 m/day

Tableau 1.5. *Problem parameters.*

2. Solution

2.1. *Question 1*

As in Exercise 1.4, x is expressed as a function of s. As seen in Exercice 1.4, the saturation profile verifies

$$\left.\begin{array}{ll} x(t>0)=\lambda(s)t & \text{for } s_0 < s \le s_i \\ s(t>0)=s_0 & \text{for } x > \lambda(s_0)t \end{array}\right\} \qquad [1]$$

In contrast with Exercise 1.4, s_0 and s_i are now located on different sides of the saturation $s_{\max}$ for which the wave celerity is maximum (since $b_{BL} = 1$, $s_{\max} = 1/2$). IF Eq. [1] was to be applied without correction, a three-valued profile would be obtained, as shown in Figure 1.

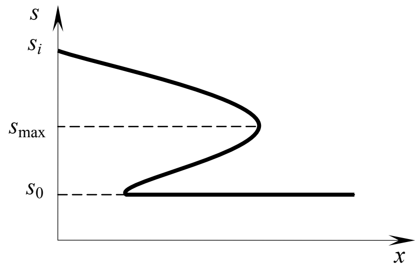

Figure 1. *Saturation profile as given by Eq. [1].*

The profile $x(s)$ has a maximum for $s = s_{\max}$, which indicates that there exists a region of space where $s(x)$ may take several values simultaneously, which is not permissible. The only permissible solution is a compound wave, the location of which is determined using the equal area rule so as to guarantee mass conservation (Figure 2).

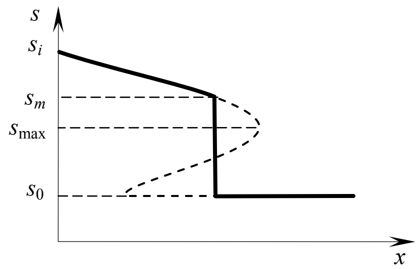

Figure 2. *Saturation profile before correction (dashed line) and after correction (solid line).*

2.2. *Question 2*

The shock in the compound wave moves at a speed c_m given by

$$F(s_m) - F(s_0) = (s_m - s_0)c_m \qquad [2]$$

where s_m is the water satruation immediately behind the shock. Since the point $s = s_m$ also belongs to the rarefaction wave, one has

$$c_m = \lambda(s_m) = \frac{\mathrm{d}F}{\mathrm{d}s}(s_m) \qquad [3]$$

Substituting Eq. [3] into Eq. [2] leads to

$$F(s_m) - F(s_0) = (s_m - s_0)\lambda(s_m) \qquad [4]$$

Eq. [4] is a nonlinear algebraic equation in s_m. Substituting the definition of the flux F into [4] leads to

$$\frac{s_m^2}{s_m^2 + (1-s_m)^2 b_{\mathrm{BL}}} - 2(s_m - s_0)\frac{(1-s_m)s_m b_{\mathrm{BL}}}{\left[s_m^2 + (1-s_m)^2 b_{\mathrm{BL}}\right]^2} = \frac{F(s_0)}{V_d} \qquad [5]$$

Eq. [5] is solved iteratively. Te data in Table 1.5 yields $s_m = 0{,}675$ and $c_m = 1{,}39$ m/day (see the accompanying spreadsheet).

2.3. *Question 3*

The solution is found exactly in the same way as in Exercise 1.4.