Exercise 4.1

The Saint Venant equations

This document provides an outline for the solution of Exercise 4.1 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Solve the Riemann problem for the Saint-Venant equations in a rectangular channel for the following initial data (note the symmetry in the initial state)

$$\left.\begin{aligned} h_L &= h_R \\ u_L &= -u_R \end{aligned}\right\} \qquad [4.63]$$

Show that the nature of the waves in the solution depends on the sign of u_L. Provide a complete description of the solution in each case (nature of the waves, expressions for the water level and velocity in each of the regions). Show that a dry zone may appear if u_L is negative, smaller than a given threshold value to be determined. Explain how the symmetry of this Riemann problem may be used to solve boundary value problems, such as the sudden closure of a valve at the upstream and downstream ends of a canal.

2. Solution

It is first recalled that the solution of the Riemann problem for the Saint Venant equations is made of an intermediate region of constant state separated from the left and right state by two waves.

2.1. *Preliminary remark : symmetry considerations*

Symmetry considerations allow the following properties to be identified for the solution :

– the Riemann problem remains unchanged if the direction of the x-axis is changed. Consequently, the flow velocity in the intermediate region of constant state verifies $u^* = -u^*$, that is

$$u^* = 0 \qquad [1]$$

– the types of the leftward and rightward waves are identical. Their propagation speed is identical, only the sign changes. Therefore the solution is determined completely provided that e.g. the leftward wave has been determined.

2.2. *Case* $u_L > 0$

In the case $u_L > 0$, both the heuristic approach and the Riemann invariants lead to the conclusion that the celerity $u^* - c^*$ in the intermediate region of constant state is larger than the celerity $u_L - c_L$ in the left state. The leftward wave should then be a shock wave. In this case the Rankine-Hugoniot relationships are applicable

$$\left.\begin{aligned} u_L h_L &= (h_L - h^*)c_s \\ u_L^2 h_L + \frac{g}{2}(h_L^2 - h^{*2}) &= u_L h_L c_s \end{aligned}\right\} \qquad [2]$$

where c_s is the propagation speed of the shock. Note that c_s is negative.

Eliminating the shock speed c_s from Eqs. [2], multiplying by $h_L - h^*$ leads to

$$u_L^2 h_L (h_L - h^*) + \frac{g}{2}\left(h_L^2 - h^{*2}\right)^2 (h_L - h^*) = (u_L h_L)^2 \qquad [3]$$

The function $f(h)$ is introduced

$$f(h) = \left[u_L^2 h_L + \frac{g}{2}\left(h_L^2 - h^2\right)\right](h_L - h) \qquad [4]$$

Then h^* is the value of h for which f satisfies the following condition

$$f(h^*) = (u_L h_L)^2 \qquad [5]$$

It is easy to check that

– f takes a zero value for $h = h_L$;

– its derivative is positive for $h < h_1$ and $h > h_2$, with h_1 negative and h_2 larger than h_L.

Consequently, h^* must be sought either within the interval $[h_1, h_L[$ or for values larger than h_2. The first equation [2] indicates however that h^* is larger than h_L because c_s is negative. The only physically permissible root is therefore larger than h_2. Solving Eq. [5] using an iterative technique such as the Newton-Raphson method yields the solution. Since the Riemann invariants provide fairly reasonable approximations of the jump relationships, it is advised that the iterations should be initialised using these Riemann invariants

$$c^* = \frac{u_L}{2} + c_L \qquad [6]$$

and $h^* = c^2/g$.

2.3. *Case $u_L < 0$*

In this case the wave is a rarefaction wave. The Riemann invariants are applicable across the wave and Eq. [6] holds. Note that the propagation speed c^* becomes zero (and so does the water depth h^*) if u_L is equal to the following value

$$u_L^{dry} = -2c_L \quad [7]$$

Any value of u_L smaller than this limit value leads to the creation of a drying zone (Figure 1).

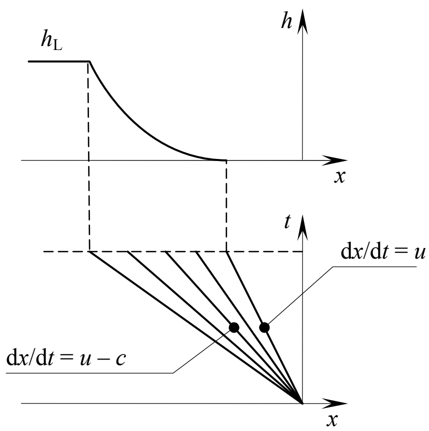

Figure 1. *Appearance of a drying zone. Sketch in the physical space (top) and in the phase space (bottom).*

The tail of the drying zone moves at a speed $u - c$, with

$$\left.\begin{aligned} u + 2c &= u_L + 2c_L \\ c &= 0 \end{aligned}\right\} \quad [8]$$

The first equation stems directly from the second Rieman invariant $u + 2\,c$. The secon equation accounts for the assumption of a dry point. Eqs. [8] yield

$$u = u_L + 2c_L < 0 \quad [9]$$

2.4. *Practical consequences of symmetry properties*

As mentioned in 2.1, any situation where the discharge is zero at a given point may be obtained from a Riemann problem where the left and right states are symmetrical. This property is sometimes used in finite volume methods when no-flow boundary conditions must be prescribed.