Exercise 6.1 (1)

Finite difference methods for scalar laws

This document provides an outline for the solution of Exercise 6.1 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE. The application to Exercise 1.1 is detailed hereafter.

1. Problem

Check the conclusions of Exercises 1.1 to 1.5 using finite difference methods. The following methods are advised:

– a characteristic-based method,

– an upwind scheme (conservative version),

– Preissmann's scheme,

– a TVD scheme.

2. Solution

2.1. *The MOC*

2.1.1. *Calculation of the flow velocity*

The characteristic form [1.67] of the inviscid Burgers equation is recalled

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \quad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u$$

Integrating the differential equation along the characteristic yields

$$u_i^{n+1} = u_\mathrm{A} \qquad [1]$$

where A is the foot of the characteristic passing at the point (i, n) in the phase space (Figure 1). The value of u_A is intrpolated from the values at $i - 1$ and $i + 1$. The first-order interpolation for a positive advection velocity is recalled

$$u_i^{n+1} = \mathrm{Cr}\, u_{i-1}^n + (1 - \mathrm{Cr})u_i^n \qquad [2]$$

while the second-order interpolation gives

$$u_i^{n+1} = \frac{\mathrm{Cr}}{2}(\mathrm{Cr}+1)u_{i-1}^n + (1-\mathrm{Cr}^2)u_i^n + \frac{\mathrm{Cr}}{2}(\mathrm{Cr}-1)u_{i+1}^n \quad [3]$$

where the Courant number Cr is defined as

$$\mathrm{Cr} = \frac{\Delta t}{\Delta x} u_i^{n+1/2} \quad [4]$$

where $u_i^{n+1/2}$ is the average value of u along the characteristic.

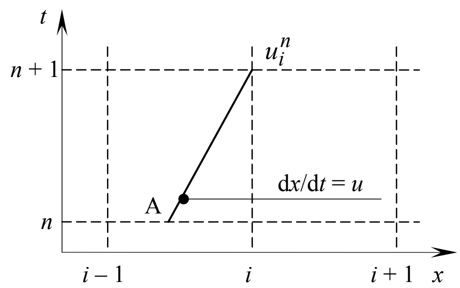

Figure 1. *Definitin sketch for the MOC.*

A more general formula may be proposed for $u_i^{n+1/2}$

$$u_i^{n+1/2} = (1-\theta)u_{i-1}^n + \theta u_i^n \quad [5]$$

where θ is a centring parameter between 0 and 1. If $\theta = 0$ the speed of the characteristic is taken from the previous time step at the point located immediately upstram. If $\theta = 1$, the celerity of the current point at the previous time step is used.

2.1.2. *Calculaytion of the density*

The characteristic form of the continuity equation is recalled

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\partial u}{\partial x}\rho \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \quad [6]$$

Integrating [6] explicitly gives

$$\rho_i^{n+1} = \left(1 - \frac{\partial u}{\partial x}\Delta t\right)\rho_\mathrm{A} \quad [7]$$

The derivative of u with respect to x is estimated using an explicit upwind approach

$$\frac{\partial u}{\partial x} \approx \frac{u_i^n - u_{i-1}^n}{\Delta x} \quad [8]$$

and the value of ρ at A is interpolated in the same way as that of u. A first-order interpolation gives

$$\rho_\mathrm{A} = \mathrm{Cr}\ \rho_{i-1}^n + (1-\mathrm{Cr})\rho_i^n \quad [9]$$

while a second-order interpolation leads to

$$\rho_A = \frac{\text{Cr}}{2}(\text{Cr}+1)\rho_{i-1}^n + (1-\text{Cr}^2)\rho_i^n + \frac{\text{Cr}}{2}(\text{Cr}-1)\rho_{i+1}^n \qquad [10]$$

2.1.3. *Test case*

The behaviour of the analytical and numerical solutions at $t = 5$ s and $t = 10$ s is illustrated by Figure 2. The initial condition consists of a triangular velocity profile and a uniform density.

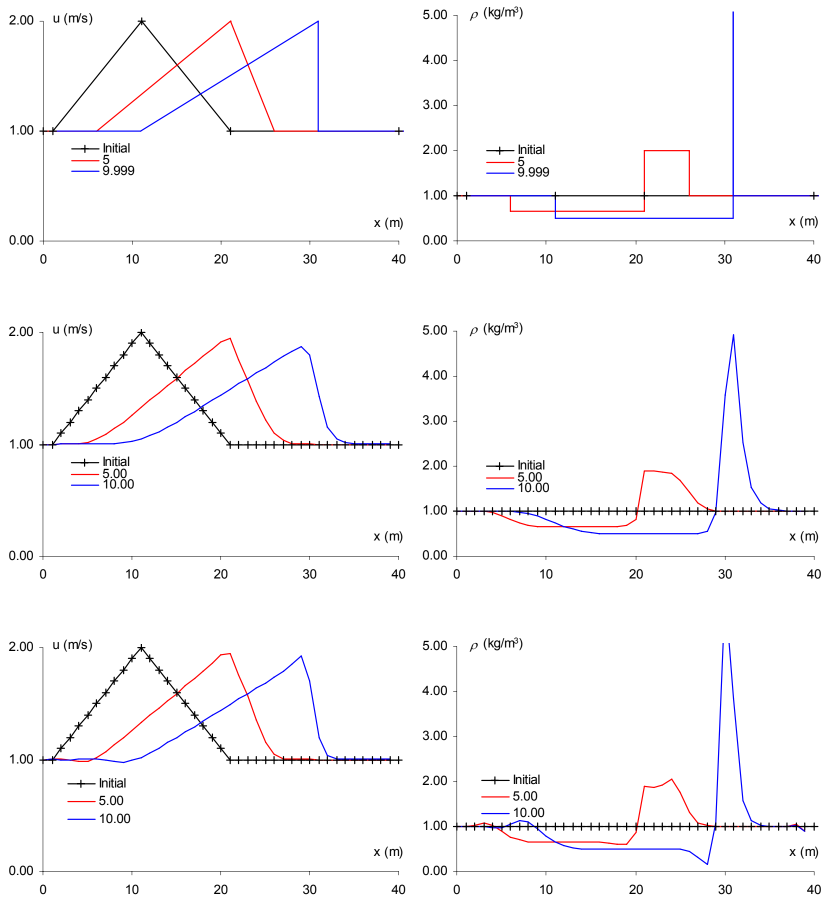

Figure 2. *Solution of the inviscid Burgers equation. Velocity profiles (left) and density profiles (right) for the analytical solution (top), the first-order MOC (middle) and the second-order MOC (bottom).*

Note that $t = 10$ sec. is the time at which the shock appears (see Exercises 3.2). A cell size $\Delta x = 1$ m and a time step $\Delta t = 0.5$ s are used, with a centring parameter $\theta = 1/2$.

The following remakrs are made.

– With a time step $\Delta t = 0.5$ s, the maximum Courant number is obtained for $u = 2$ m/s. It is equal to unity. This numerical value corresponds to the stability limit of the explicit MOC. It is also the value for which the analytical solution is obtained. However, since $\theta = ½$ is used in the simulation, Cr is equal to unity only if two consecutive calculation points take the value $u = 2$ m/s. This is not the case in the present initial conditions. For the maximum value of 2 m/s to be preserved, a coefficient $\theta = 0$ should be used. These conclusions can be checked from the calculation spreadsheet (see http://vincentguinot.free.fr/waves/ex611.xls);

– numerical diffusion exerts a visible influence on the numerical solution obtained with the first-order MOC, with fronts becoming smoother in time. Note that numerical diffusion is maximum for Cr = ½, which corresponds to $u = 1$ m/s (see Appendix B of the book) ;

– numerical dispersion can be observed in the numerical solution given by the second-order MOC. It is accountable for the oscillations observed in the profiles ;

– the numerical solution obtained from the MOC is very likely not to satisfy conservation at times $t > 10$ s because the solution becomes discontinuous (see Chapter 6 for detailed considerations on conservation issues).

2.2. *Conservative upwind scheme*

2.2.1. *Option 1 : u and ρ are computed independently*

A first option consists in solving the conservation form in ρ (Eq. [1.62]) and u (Eq. [1.69]) separately. The conservative upwind discretization yields

$$\left.\begin{aligned} u_i^{n+1} &= u_i^n + \frac{\Delta t}{\Delta x}\frac{(u_{i-1}^n)^2 - (u_i^n)^2}{2} \\ \rho_i^{n+1} &= \rho_i^n + \frac{\Delta t}{\Delta x}\left[(u\rho)_{i-1}^n - (u\rho)_i^n\right] \end{aligned}\right\} \qquad [11]$$

2.2.2. *Option 2 : u and ρ are computed simultaneously*

As seen in Chapter 3, Eq. [1.69] is not equivalent to Eq. [1.63] when the solution is discontinuous because the conserved variables are not identical. The second option consists in working with the conserved variables ρ and ρu. The resulting discretization is

$$\left.\begin{aligned} \rho_i^{n+1} &= \rho_i^n + \frac{\Delta t}{\Delta x}\left[(u\rho)_{i-1}^n - (u\rho)_i^n\right] \\ (\rho u)_i^{n+1} &= (\rho u)_i^n + \frac{\Delta t}{\Delta x}\left[(\rho u^2)_{i-1}^n - (\rho u^2)_i^n\right] \end{aligned}\right\} \qquad [12]$$

Substituting the first equation [12] into the second leads to

$$\left.\begin{aligned} \rho_i^{n+1} &= \rho_i^n + \frac{\Delta t}{\Delta x}\left[(u\rho)_{i-1}^n - (u\rho)_i^n\right] \\ u_i^{n+1} &= \frac{(\rho u)_i^n}{\rho_i^{n+1}} + \frac{1}{\rho_i^{n+1}}\frac{\Delta t}{\Delta x}\left[(\rho u^2)_{i-1}^n - (\rho u^2)_i^n\right] \end{aligned}\right\} \qquad [13]$$

The density ρ_i^{n+1} is computed first and used in the calulation of u_i^{n+1}.

2.2.3. *Application*

The test presented in Subsection 2.1.3 is repeated, with the same values for Δt et and Δx.

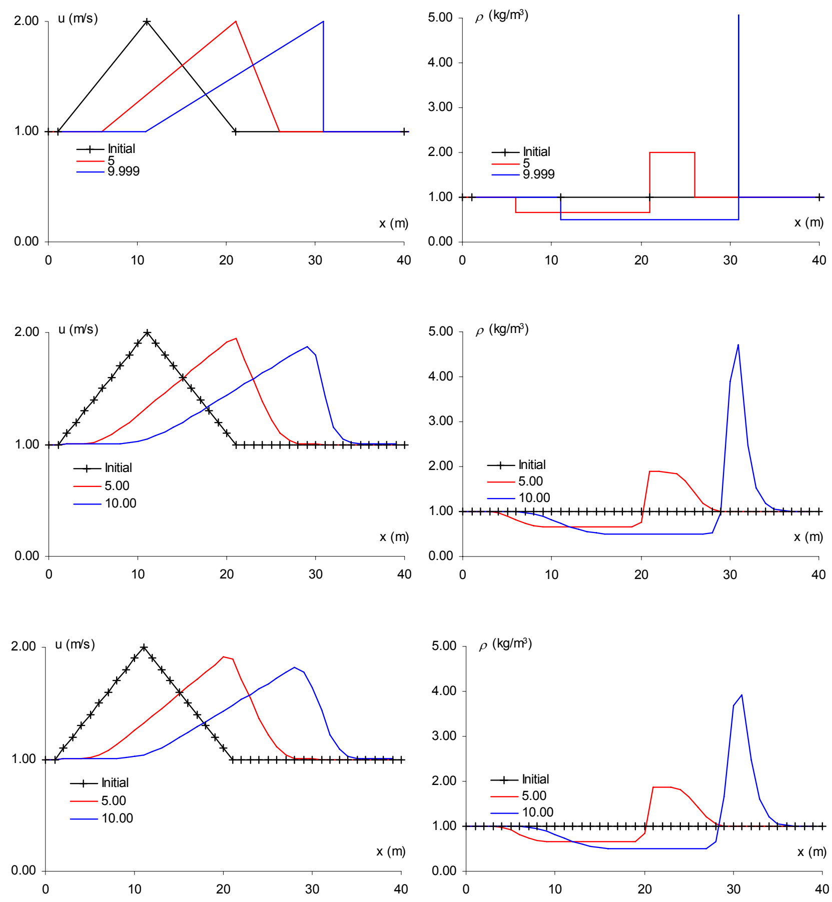

Figure 3. *Solution of the inviscid Burgers equation. Velocity profiles (left) and density profiles (right) for the analytical solution (top), the upwind scheme with option 1 (middle) and the upwind scheme with option 2 (bottom).*

The second option can be seen to be slightly more diffusive than the first. This could have been expected because u is computed using the value of ρ, that is also subjected to numerical diffusion.

2.3. *Preissmann's scheme*

2.3.1. *Calculation of u*

The nonconservation form [1.66] is easier to solve than the conservation formes [1.63] and [1.69]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad [1.66]$$

The equation is discretized as follows

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} &\approx (1-\psi)\frac{u_{i-1}^{n+1}-u_{i-1}^{n}}{\Delta t}+\psi\,\frac{u_i^{n+1}-u_i^{n}}{\Delta t}\\
u &\approx (1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\\
\frac{\partial u}{\partial x} &\approx (1-\theta)\frac{u_i^{n}-u_{i-1}^{n}}{\Delta x}+\theta\,\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}
\end{aligned}\right\} \qquad [14]$$

The estimate of u involves only values at the time level n in order to avoid the presence of second-degree terms in u_i^{n+1} in the final equation. Substituting the approximations [14] into Eq. [1.66] leads to

$$\begin{aligned}
&(1-\psi)\frac{u_{i-1}^{n+1}-u_{i-1}^{n}}{\Delta t}+\psi\,\frac{u_i^{n+1}-u_i^{n}}{\Delta t}\\
&+\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\left((1-\theta)\frac{u_i^{n}-u_{i-1}^{n}}{\Delta x}+\theta\,\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}\right)=0
\end{aligned} \qquad [15]$$

Rearranging gives

$$\begin{aligned}
&\left\{-\frac{1-\psi}{\Delta t}-\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{1-\theta}{\Delta x}\right\}u_{i-1}^{n}\\
&+\left\{-\frac{\psi}{\Delta t}+\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{1-\theta}{\Delta x}\right\}u_i^{n}\\
&+\left\{\frac{1-\psi}{\Delta t}-\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{\theta}{\Delta x}\right\}u_{i-1}^{n+1}\\
&+\left\{\frac{\psi}{\Delta t}+\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{\theta}{\Delta x}\right\}u_i^{n+1}=0
\end{aligned} \qquad [16]$$

Solving Eq. [16] for u_i^{n+1} leads to

$$u_i^{n+1}=-\frac{a}{d}u_{i-1}^{n}-\frac{b}{d}u_i^{n}-\frac{c}{d}u_{i-1}^{n+1} \qquad [17]$$

where the coefficients a to d are given by

$$\left.\begin{aligned}
a &= -\frac{1-\psi}{\Delta t}-\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{1-\theta}{\Delta x}\\
b &= -\frac{\psi}{\Delta t}+\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{1-\theta}{\Delta x}\\
c &= \frac{1-\psi}{\Delta t}-\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{\theta}{\Delta x}\\
d &= \frac{\psi}{\Delta t}+\left[(1-\alpha)u_{i-1}^{n}+\alpha\, u_i^{n}\right]\frac{\theta}{\Delta x}
\end{aligned}\right\} \qquad [18]$$

2.3.2. *Calculation of ρ*

The conservation form [1.62] is solved

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

with the following estimates

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} &\approx (1-\psi)\frac{u_{i-1}^{n+1} - u_{i-1}^{n}}{\Delta t} + \psi \frac{u_i^{n+1} - u_i^n}{\Delta t} \\ \frac{\partial (\rho u)}{\partial x} &\approx (1-\theta)\frac{(\rho u)_i^n - (\rho u)_{i-1}^n}{\Delta x} + \theta \frac{(\rho u)_i^{n+1} - (\rho u)_{i-1}^{n+1}}{\Delta x} \end{aligned}\right\} \quad [19]$$

The discretized equation thus becomes

$$\begin{aligned} &(1-\psi)\frac{\rho_{i-1}^{n+1} - \rho_{i-1}^{n}}{\Delta t} + \psi \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} \\ &+ (1-\theta)\frac{(\rho u)_i^n - (\rho u)_{i-1}^n}{\Delta x} + \theta \frac{(\rho u)_i^{n+1} - (\rho u)_{i-1}^{n+1}}{\Delta x} = 0 \end{aligned} \quad [20]$$

Rearranging leads to

$$\begin{aligned} &\left(-\frac{1-\psi}{\Delta t} - \frac{1-\theta}{\Delta x} u_{i-1}^n\right)\rho_{i-1}^n \\ &+ \left(-\frac{\psi}{\Delta t} + \frac{1-\theta}{\Delta x} u_i^n\right)\rho_i^n \\ &+ \left(\frac{1-\psi}{\Delta t} - \frac{\theta}{\Delta x} u_{i-1}^{n+1}\right)\rho_{i-1}^{n+1} \\ &+ \left(\frac{\psi}{\Delta t} + \frac{\theta}{\Delta x} u_i^{n+1}\right)\rho_i^{n+1} = 0 \end{aligned} \quad [21]$$

Solving [21] for ρ_i^{n+1} leads to

$$\rho_i^{n+1} = -\frac{a'}{d'}\rho_{i-1}^n - \frac{b'}{d'}\rho_i^n - \frac{c'}{d'}\rho_{i-1}^{n+1} \quad [22]$$

with

$$\left.\begin{aligned} a' &= -\frac{1-\psi}{\Delta t} - \frac{1-\theta}{\Delta x} u_{i-1}^n \\ b' &= -\frac{\psi}{\Delta t} + \frac{1-\theta}{\Delta x} u_i^n \\ c' &= \frac{1-\psi}{\Delta t} - \frac{\theta}{\Delta x} u_{i-1}^{n+1} \\ d' &= \frac{\psi}{\Delta t} + \frac{\theta}{\Delta x} u_i^{n+1} \end{aligned}\right\} \quad [23]$$

2.3.3. *Application*

The previous test case is used again. The centring coefficient α in Eq. [14] is set to ½.

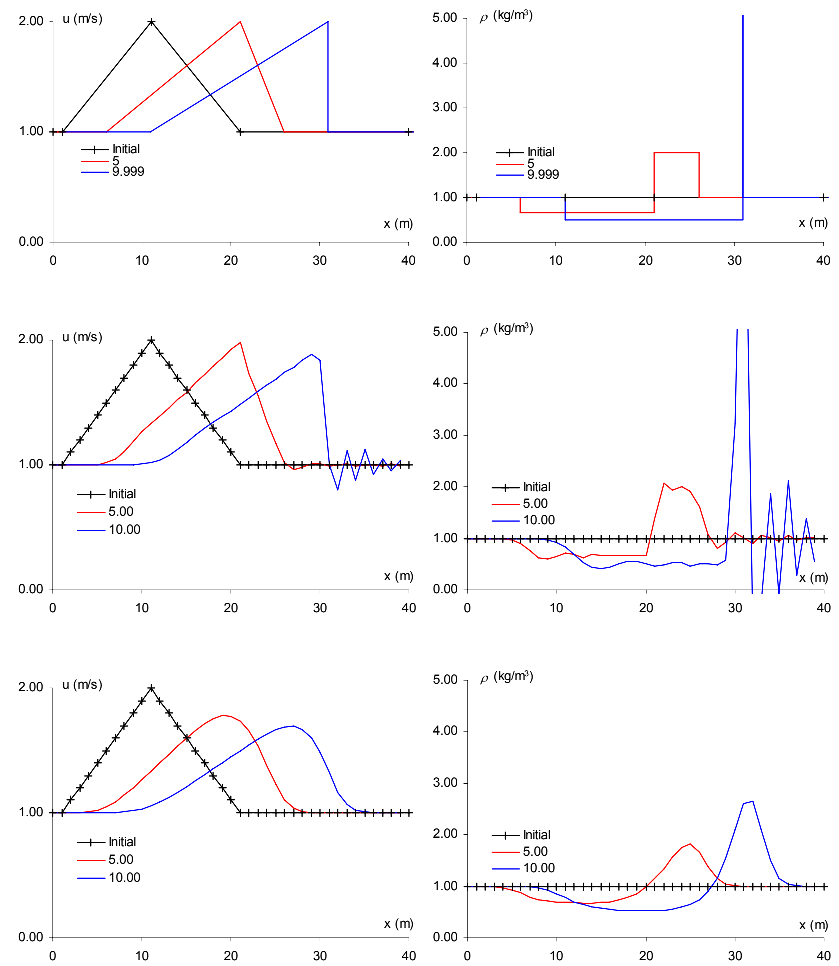

Figure 4. *Solution of the inviscid Burgers equation. Velocity profiles (left) and density profiles (right) for the analytical solution (top), the Preissmann scheme with $\theta = 1/2$ (middle) and the Preissmann scheme with $\theta = 1$ (bottom).*

Note that

– when $\theta = ½$ and $\psi = ½$ the solution is not subjected to numerical diffusion. The scheme becomes dispersive, which results in an oscillatory behaviour of the numerical solution, as shown in Figure 4 (middle) ;

– the oscillations may be eliminated by increasing θ. They are completely removed for $\theta = 1$ (Figure 4, bottom), which corresponds to the strongest possible smoothing. Acceptable results are usually obtained with $\theta = 0,65$.

2.4. *TVD scheme*

2.4.1. *Calculation of u*

The nonconservation form [1.66] is considered

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad [1.66]$$

The equation is discretized as in Subsection 6.6.2 of the book

$$u_i^{n+1} = u_i^n - (u_i^n - u_{i-1}^n)\mathrm{Cr} + \frac{\mathrm{Cr}^2 - \mathrm{Cr}}{2}\left[(u_{i+1}^n - u_i^n)\phi_{i+1/2} - (u_i^n - u_{i-1}^n)\phi_{i-1/2}\right] \quad [24]$$

where ϕ is a limiting function that obeys the criteria presented in Subsections 6.6.3 and 6.6.4. Cr is computed using an estimate $u_{i-1/2}$ of the velocity that satisfies momentum conservation (see Section 6.8 of the book)

$$u_{i+1/2} = \frac{1}{2}\frac{(u^2)_{i+1}^n - (u^2)_i^n}{u_{i+1}^n - u_i^n} = \frac{u_{i+1}^n + u_i^n}{2} \quad [25]$$

2.4.2. *Calculation of* ρ

The TVD scheme is applied to the nonconservation form [1.62] :

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = -\rho\frac{\partial u}{\partial x} \quad [26]$$

The equation is discretized as

$$\rho_i^{n+1} = \rho_i^n - (\rho_i^n - \rho_{i-1}^n)\mathrm{Cr} + \frac{\mathrm{Cr}^2 - \mathrm{Cr}}{2}\left[(\rho_{i+1}^n - \rho_i^n)\phi_{i+1/2} - (\rho_i^n - \rho_{i-1}^n)\phi_{i-1/2}\right] - \Delta t\frac{\partial u}{\partial x}\rho_{i-1}^n \quad [27]$$

The derivative of u with respect to x is estimated using an explicit upwind method

$$\frac{\partial u}{\partial x} \approx \frac{u_i^n - u_{i-1}^n}{\Delta x} \quad [28]$$

2.4.3. *Application*

The results obtained using the MC limiter are shown in Figure 5.

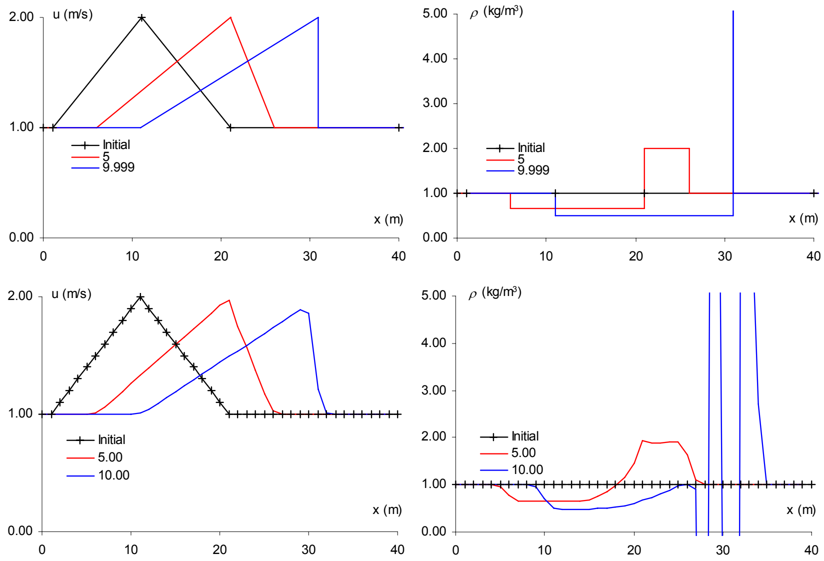

Figure 5. *Solution of the inviscid Burgers equation.Velocity profiles (left) and density profiles (right) for the analytical solution (top) and the TVD scheme with MC limiter (bottom).*