Exercise 5.1

The Doppler effect

This document provides an outline for the solution of Exercise 4.2 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider a mobile sound source that moves at a speed u smaller than the speed of sound. The frequency N of the sound is constant. Using the secant plane approach, show that the frequency N' of the sound as heard by an immobile observer is given by

$$N' = \frac{N}{(1 - M\cos\theta)} \qquad [5.85]$$

where M is the Mach number and θ is the angle between the velocity vector of the source and the direction of the straight line drawn from the observer to the source (Figure 5.12).

This phenomenon is known as the Doppler effect.

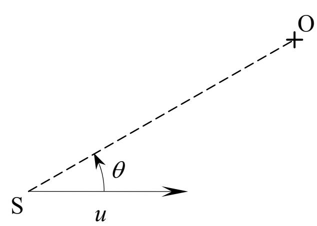

Figure 5.12. *Mobile sound source.*

2. Solution

Consider the secant plane that contains both the soud source S and the observer O in the phase space. The speed of the source in this plane is $u\cos\theta$. The sound wave issued from the source at the time $t = 0$ originates from the point $x = 0$

(Figure 1). After one soud period $T = 1/N$ the wave reaches the point A, the abscissa of which is cT.

At the time T the source is located at the point S_T (Figure 1). The abscissa of the point is $x_T = u\ t \cos\theta$. The sound wave starting from this point at $t = T$ travels at the speed of sound and reaches the point B (the abscissa of which is identical to that of A) after a time interval T'.

The movement of the source and the propagation of the sound waves are sketched in Figure 1. The characteristic $\mathrm{d}x/\mathrm{d}t = u\cos\theta + c$ moves in the direction of the observer, while the characteristic $\mathrm{d}x/\mathrm{d}t = u\cos\theta - c$ travels in the opposite direction.

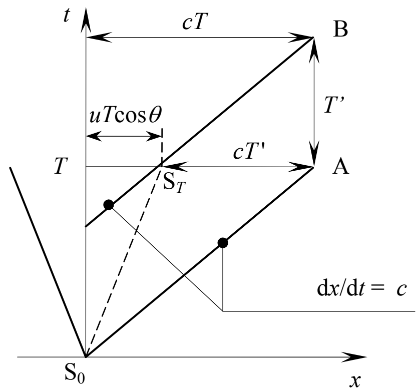

Figure 1. *Definition sketch of the wave propagation problem in the secant plane* (SO).

Consequently the following relationship holds

$$x_B = x_T + cT' = uT\cos\theta + cT' \tag{1}$$

Moreover

$$x_A = cT \tag{2}$$

Since $x_A = x_B$, one has

$$uT\cos\theta + cT' = cT \tag{3}$$

Dividing by c leads to

$$T' = (1 - M\cos\theta)\,T \tag{4}$$

The frequency is given as the inverse of the period

$$N' = \frac{N}{1 - M\cos\theta} \tag{5}$$

Which is exactly Eq. [5.85]. A first-order series expansion yields the following approximation

$$N' \approx (1 + M\cos\theta)\,N \tag{6}$$

When $M < 1$, a source moving in the direction of the observer gives a frequencyN' higher than N because $\cos\theta$ is positive. Conversely, if the source moves away from the observer, the frequency N' is lower than N.

Note that the maximum frequency is perceived when $\cos\theta = 1$, that is, in the situation where the source is moving straight at the observer. The minimum frequency is heard if the source is moving straight away from the observer because $\cos\theta = -1$

$$\left.\begin{aligned} N'_{\max} &= \frac{N}{1-M} \approx (1+M)\,N \\ N'_{\max} &= \frac{N}{1+M} \approx (1-M)\,N \end{aligned}\right\} \qquad [7]$$

The ratio of the extreme frequencies is given by

$$\frac{N'_{\max}}{N'_{\min}} = \frac{1+M}{1-M} \qquad [8]$$

For instance, an ambulance moving at 50 km/hr (typical speed limit in urban areas, u = 13,9 m/s) is characteriszed by a Mach number M = 0,04, which leads to a rfequency ratio of 1,085. Since multiplying the frequency by $2^{1/12}$ leads to raising the tone by half a step, an immobile observer perceives a drop of approximately three quarters of a step in the tone of the siren when passed pby the ambulance. If the ambulance moves at 90 km/hr, the tone is heard to drop by 5/4 of a step. A drop of two steps corresponds to a vehicle moving at 140 km/hr.