Exercise 3.4

The Saint Venant equations

This document provides an outline for the solution of Exercise 3.4 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider the channel of Exercise 3.1, where the Saint Venant equations are to be applied instead of the kinematic wave approximation.

1) The initial water depth is assumed to be uniformly equal to 1 m. Compute the celerities of the waves for the hydraulic parameters given in Table 2.1. Show that the flow regime depends on the slope. Provide the expression of the slope S_c for which the flow is critical.

2) A perturbation $\Delta h = 1$ m in the water level appears instantaneously at the upstream end of the channel. This triggers a moving bore that propagates to the right. Assuming that the flow regime is subcritical, provide the expression satisfied by the variation ΔQ in the discharge. Carry out the numerical approximation for $S_0 = 10^{-3}$.

Symbol	Meaning	Value
b	Channel width	10 m
g	Gravitational acceleration	9.81 m s^{-2}
K_{Str}	Strickler coefficient	40
S_0	Channel bed slope	0.1 %, 1% and 5%

Table 2.1. *Problem parameters.*

2. Solution

2.1. *Question 1*

Under the assumption of a uniform flow, the wave celerities are given by (see Exercise 2.5) :

$$\left.\begin{aligned}\lambda^{(1)} &= K_{\mathrm{Str}} S_0^{1/2} h^{2/3} - (gh)^{1/2} \\ \lambda^{(2)} &= K_{\mathrm{Str}} S_0^{1/2} h^{2/3} + (gh)^{1/2}\end{aligned}\right\} \qquad [1]$$

The behaviour of the celerities is illustrated in Figure 1 for slopes 10^{-3}, 10^{-2} and 5×10^{-2}.

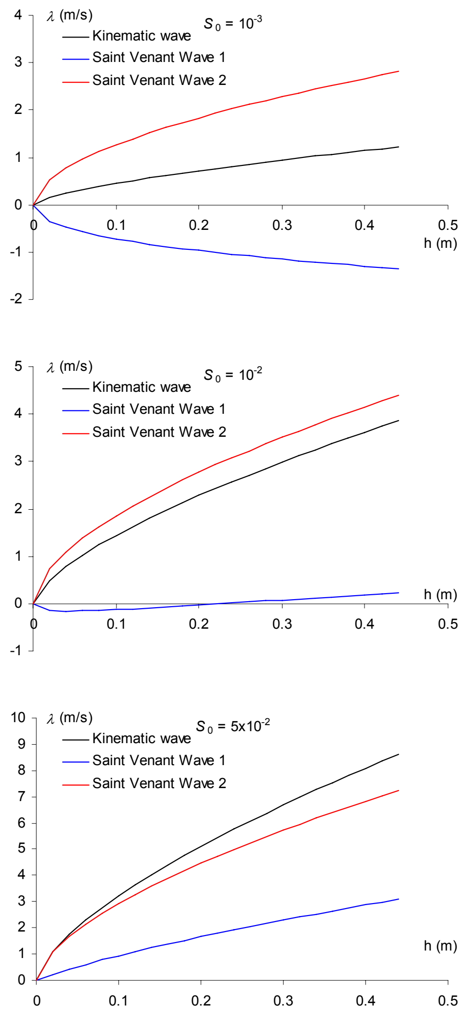

Figure 1. *Wave celerities as a function of h for various values of the bed slope.*

The celerity $\lambda^{(1)}$ is equal to sero for a water depth h_c given by

$$h_c = \left(\frac{g}{K_{\mathrm{Str}}^2 S_0}\right)^3 \qquad [2]$$

The flow regime is subcritical when h is smaller than h_c and supercritical otherwise. The transition between subcritical and supercritical flow for a fixed depth occurs for a slope S_c given by

$$S_c = \frac{g}{K_{\mathrm{Str}}^2 h^{1/3}} \qquad [3]$$

The parameters in the Table leads to a critical slope $S_c = 6\times10^{-3}$ for $h = 1$ m. Both u and c are equal to 3.1 m/s.

2.2. *Question 2*

When the water depth rises from $h = h_0$ to $h = h_1 = h_0 + \Delta h$, the discharge rises from Q_0 to Q_1. The shock wave propagates at a speed c_s. Both Q_1 and c_s are determined using the jump relationships

$$\left.\begin{aligned} & Q_1 - Q_0 = (h_1 - h_0)bc_s \\ & \frac{Q_1^2}{bh_1} + \frac{bg}{2}h_1^2 - \frac{Q_0^2}{bh_0} - \frac{bg}{2}h_0^2 = (Q_1 - Q_0)c_s \end{aligned}\right\} \quad [4]$$

Eliminating c_s from the equations leads to

$$\frac{Q_1^2}{bh_1} + \frac{bg}{2}h_1^2 - \frac{Q_0^2}{bh_0} - \frac{bg}{2}h_0^2 = \frac{(Q_1 - Q_0)^2}{b\Delta h} \quad [5]$$

This is a second-degree equation in Q_1 that can be rewritten as

$$\left.\begin{aligned} & AQ_1^2 + BQ_1 + C = 0 \\ & A = \frac{1}{h_1} - \frac{1}{\Delta h} \\ & B = \frac{2Q_0}{\Delta h} \\ & C = \left(\frac{1}{\Delta h} - \frac{1}{h_0}\right)Q_0^2 + \frac{b^2 g}{2}(h_1^2 - h_0^2) \end{aligned}\right\} \quad [6]$$

with roots

$$Q_1 = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2A} \quad [7]$$

Note that A is negative. Since AC is negative, the roots [7] have opposite signs. Only the positive root is meaningful, that is

$$Q_1 = \frac{-B - (B^2 - 4AC)^{1/2}}{2A} \quad [8]$$

The data in Table 2.1 leads to $Q_1 = 85$ m^3 s^{-1} and $c_s = 7.2$ m s^{-1}. The celerity $\lambda^{(2)}$ behind the shock is then $\lambda_1^{(2)} = 8.7$ m s^{-1}.