Exercise 3.2

The kinematic wave

This document provides an outline for the solution of Exercise 3.2 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider the channel of Exercise 3.1, with the same geometry and initial conditions. The water depth a the upstream end of the channel is now assumed to increase linearly from 1 m to 1.25 m between $t = 0$ and $t = 100$ s, and to decrease linearly from 1.25 m to 1 m between $t = 100$ s and $t = 200$ s.

1) Assuming that the kinematic wave approximation is applicable, provide the expression of the time t_d at which the solution becomes discontinuous. Compute t_d and the location of the shock at $t = t_d$ from the parameters in Table 2.1.

2) Plot the water level profile at $t = 150$ s, 300 s, 450 s and 600 s. *N.B.*: it is advised to express both h and x as functions of the time t_L at which the characteristic leaves the left-hand end of the channel.

Symbol	Meaning	Value
b	Channel width	10 m
g	Gravitational acceleration	9.81 m/s^2
K_{Str}	Strickler coefficient	40
S_0	Channel bottom slope	0.1 %, 1%, 5%

Table 2.1. *Parameters for Exercise 3.2.*

2. Solution

2.1. *Question 1*

Under the wide channel approximation, the celerity λ of the kinematic wave is given by

$$\lambda = \frac{5}{3} K_{Str} S_0^{1/2} h^{2/3} \qquad [1]$$

When the water depth increases, the wave celerity increases and the crest of the wave travels faster than the bottom. The head of the wave becomes steeper, while the tail becomes smoother. At the time t_d the water depth profile becomes vertical for at least one value of x : the shock wave appears. The wave celerity being larger on the left-hand side of the shock than the shock speed by definition, the crest of the wave eventually catches up the shock (Figure 1).

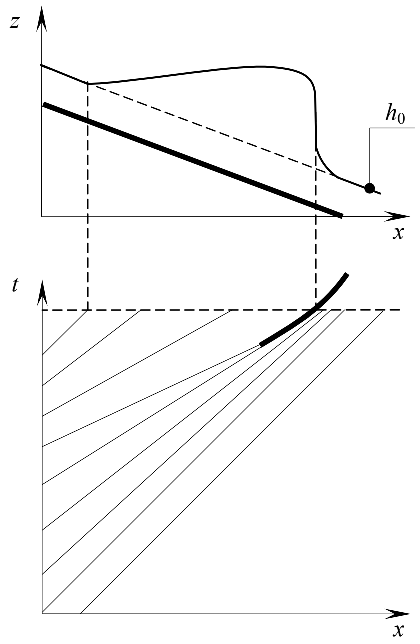

Figure 1. *Kinematic wave. Shock wave resulting from a rise in the water level. Representation in the physical space (top) and in the phase space (bottom). The trajectory of the shock is represented by a thick line in the phase space.*

The time t_d corresponds to an infinite value of $\partial h / \partial x$. To do so, the kinematic wave equation is transformed into an equation in $\partial h / \partial x$. The nonconservation form is recalled

$$\frac{\partial h}{\partial t} + \lambda \frac{\partial h}{\partial x} = 0 \qquad [2]$$

Differentiating [2] with respect to x leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial h}{\partial x}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = -\frac{\partial \lambda}{\partial x}\frac{\partial h}{\partial x} \qquad [3]$$

This leads to the following characteristic form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial x}\right) = -\frac{\partial \lambda}{\partial x}\frac{\partial h}{\partial x} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \qquad [4]$$

The derivative of λ with respect to x is rewritten as

$$\frac{\partial \lambda}{\partial x} = \frac{\mathrm{d}\lambda}{\mathrm{d}h}\frac{\partial h}{\partial x} \qquad [5]$$

and [4] becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial x}\right) = -\frac{\mathrm{d}\lambda}{\mathrm{d}h}\left(\frac{\partial h}{\partial x}\right)^2 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \tag{6}$$

Since h is constant along the characteristic lines, $\mathrm{d}\lambda/\mathrm{d}h$ is also constantt. The solution of Eq. [6] is the following

$$\left(\frac{\partial h}{\partial x}\right)^{-1}(t) = \left(\frac{\partial h}{\partial x}(t_\mathrm{L})\right)^{-1} + (t - t_\mathrm{L})\frac{\mathrm{d}\lambda}{\mathrm{d}h}(t_\mathrm{L}) \quad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \tag{7}$$

where t_L is the time at which the characteristic enters the domain at the left-hand boundary. The derivative $\partial h / \partial x$ becomes infinite when $(\partial h / \partial x)^{-1}$ becomes zero. This corresponds to a time t_d

$$t_d = \min\left[t_\mathrm{L} - \left(\frac{\partial h}{\partial x}(t_\mathrm{L})\frac{\mathrm{d}\lambda}{\mathrm{d}h}(t_\mathrm{L})\right)^{-1}\right] \tag{8}$$

note that $\partial h / \partial x$ is unknown at the left-hand boundary. The known quantity is $\partial h / \partial t$, that can be computed from $\partial h / \partial x$ using

$$\frac{\partial h}{\partial x} = -\frac{1}{\lambda}\frac{\partial h}{\partial t} \tag{9}$$

Eq. [8] thus becomes

$$t_d = \min\left[t_\mathrm{L} + \left(\frac{\partial h}{\partial t}(t_\mathrm{L})\frac{\mathrm{d}\lambda}{\mathrm{d}h}(t_\mathrm{L})\right)^{-1}\lambda(t_\mathrm{L})\right] \tag{10}$$

From the boundary conditions between $t = 0$ et $t = T$, one has

$$h(t_\mathrm{L}) = h_0 + \frac{h_1 - h_0}{T}t_\mathrm{L} \tag{11}$$

where h_1 is the maximum water depth (here, 1.25 m) and T is the rising time. Moreover

$$\frac{d\lambda}{dh} = \frac{2}{3}\frac{\lambda}{h} \tag{12}$$

Consequently

$$t_d = \min\left(\frac{5}{2}t_\mathrm{L} + \frac{3}{2}\frac{h_0}{h_1 - h_0}T\right) \tag{13}$$

t_d is minimum for $t_\mathrm{L} = 0$. Then

$$t_d = \frac{3}{2} \frac{h_0}{h_1 - h_0} T \qquad [14]$$

Note that this expression does involves neither the bottom slope nor the Strickler coefficient. The numerical values on Table 2.1 yield t_d = 600 s.

2.2. *Question 2*

The equation of the characteristic issued from $x = 0$ at $t = t_L$ is

$$\frac{dx}{dt} = \lambda(h(t_L)) \qquad [15]$$

Consequently, the sets of values (x, h) form families of curves with parameter t_L

$$\left.\begin{aligned} x(t_L, t) &= \lambda(h(t_L))\, t \\ h(t_L, t) &= h(t_L) \end{aligned}\right\} \qquad [16]$$

The accompanying spreadsheet allows the depth profile to be computed at various times. The profile $h(x)$ at t = 600 s for $S_0 = 10^{-3}$ is plotted in Figure 2.

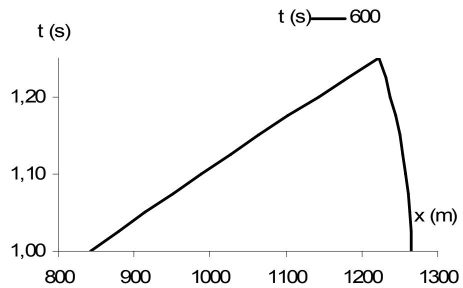

Figure 2. *Water depth profile at t = 600 s for a bottom slope 0.1 %.*