Exercise 2.4

The Saint Venant equations

This document provides an outline for the solution of Exercise 2.4 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider a rectangular channel, the width of which decreases from x_1 to x_2 and increases from x_2 to x_3 (Figure 2.24). Assuming steady state, negligible friction and bottom slope, show that

1) if the flow is subcritical ($u < c$) upstream of the narrowing and supercritical ($u > c$) downstream of it, critical conditions ($u = c$) can be reached only at the narrowest point, $x = x_2$,

2) if the flow is subcritical everywhere in the channel, the water depth reaches a minimum value at $x = x_2$,

3) if the flow is supercritical everywhere in the channel, the water depth reaches a maximum value at $x = x_2$.

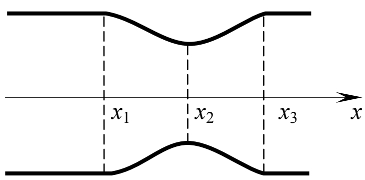

Figure 2.24. *Free surface flow in a channel with a local section narrowing.*

2. Solution

The conservation form [2.117] of the equations is used. The vector form is developed for the sake of clarity and the assumption of zero slope and negligible friction are used

$$\left.\begin{aligned}&\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0\\&\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}+\frac{P}{\rho}\right)=I_p\end{aligned}\right\}\tag{1}$$

Steady state leads to

$$\left.\begin{aligned}&\frac{\partial Q}{\partial x}=0\\&\frac{\partial}{\partial x}\left(\frac{Q^2}{A}+\frac{P}{\rho}\right)=I_p\end{aligned}\right\}\tag{2}$$

Moreover, the channel being rectangular, the terms in Eq. [2] are given by

$$\left.\begin{aligned}&\frac{P}{\rho}=b\frac{gh^2}{2}\\&I_p=\frac{gh^2}{2}\frac{\partial b}{\partial x}\\&\frac{Q^2}{A}=\frac{Q^2}{bh}\end{aligned}\right\}\tag{3}$$

where b is the width of the channel. Substituting the first equation [2] into the second equation, using [3] leads to the following equation

$$-\frac{Q^2}{bh^2}\frac{\partial h}{\partial x}-\frac{Q^2}{b^2h}\frac{\partial b}{\partial x}+bgh\frac{\partial h}{\partial x}+\frac{gh^2}{2}\frac{\partial b}{\partial x}=\frac{gh^2}{2}\frac{\partial b}{\partial x}\tag{4}$$

Simplifying and dividing by b gives

$$(c^2-u^2)\frac{\partial h}{\partial x}=\frac{u^2h}{b}\frac{\partial b}{\partial x}\tag{5}$$

At the narrowing $\partial b/\partial x=0$. Consequently, either $u=c$ (critical point) or h takes its minimum/maximum value. In the latter case

− if the flow is subcritical : $u<c$. Then the sign of $\partial h/\partial x$ is the same as that of $\partial b/\partial x$: h is minimum at the narrowing ;

− if the flow is supercritical : $u>c$. then the sign of $\partial h/\partial x$ is the opposite of that of $\partial b/\partial x$: h is maximum at the narrowing.