Exercise 2.3

Water hammer

This document provides an outline for the solution of Exercise 2.2 provided in the book *Wave Propagation in Fluids*, author V. Guinot, Publisher ISTE.

1. Problem

Consider a horizontal pipe, the cross-sectional area is the following function of the longitudinal coordinate x (Figure 2.23).

– For $x < x_1$ the section is constant, equal to A_1.

– For $x > x_2 > x_1$ the section is constant, equal to A_2

– For $x_1 < x < x_2$ the section varies continuously from A_1 to A_2.

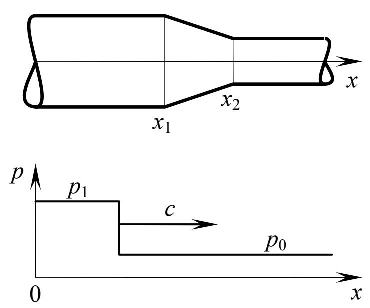

Figure 2.23. *Propagation of a pressure wave in a pipe with variable cross-sectional area.*

A_2 may be larger or smaller than A_1. The celerity c is the same all along the pipe. The effects of friction are assumed to be negligible. The water is initially flowing with a uniform pressure p_0 and a uniform discharge Q_0. At $t = 0$, the pressure at the left-hand end of the pipe changes instantaneously from p_0 to p_1. The resulting pressure discontinuity propagates to the right at the constant celerity c.

1) Provide the expression of the discharge Q_1 on the left-hand side of the pressure wave before the pressure wave reaches the abscissa x_1.

2) Provide the expression for the pressure p_2 and the discharge Q_2 when the pressure wave reaches the abscissa x_2. What is the effect of a narrowing on the pressure transient? What is the effect of a widening?

2. Solution

2.1. *Question 1*

See Exercises 2.1 et 2.2

$$Q_1 - Q_0 = \frac{A_1}{\rho c_1}(p_1 - p_0) \quad [1]$$

2.2. *Question 2*

The variations in the pressure p and the discharge Q are sought along the characteristic $\mathrm{d}x/\mathrm{d}t = +c$. The pressure and the discharge immediately behind the characteristic (i.e. on the left-hand side) are denoted by p and Q, while the pressure and discharge on the right-hand side of the characteristic are denoted by p_0 and Q_0 (Figure 1).

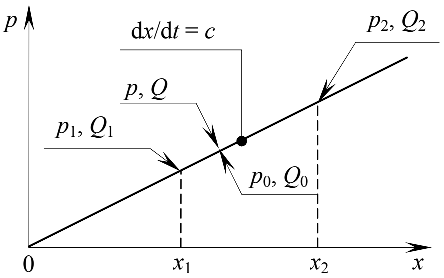

Figure 1. *Wave travelling to the right. Sketch in the phase space.*

The cross-sectional area A being variable in space, the general formulation [2.79] must be used along $\mathrm{d}x/\mathrm{d}t = +c$

$$\mathrm{d}p + \frac{\rho c}{A}\mathrm{d}Q = 0 \quad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = c \quad [2]$$

The discharge Q and the pressure p are related to p_0 and Q_0 via the characteristic $\mathrm{d}x/\mathrm{d}t = -c$

$$p = p_0 + \frac{\rho c}{A}(Q - Q_0) \quad [3]$$

where A depends on x between x_1 et x_2. Differentiating [3] leads to

$$\mathrm{d}p = \frac{\rho c}{A}\mathrm{d}Q - \frac{\rho c(Q - Q_0)}{A^2}\mathrm{d}A \quad [4]$$

Substituting Eq. [4] into Eq. [2] gives

$$2\frac{\rho c}{A}\mathrm{d}Q = \frac{\rho c(Q - Q_0)}{A^2}\mathrm{d}A \quad [5]$$

Eq. [5] is simplified into

$$2\frac{\mathrm{d}(Q-Q_0)}{Q-Q_0}=\frac{dA}{A} \qquad [6]$$

Intgrating [6] between x_1 and x_2 yields

$$\frac{Q_2-Q_0}{Q_1-Q_0}=\left(\frac{A_2}{A_1}\right)^{1/2} \qquad [7]$$

which gives

$$Q_2-Q_0=\left(\frac{A_2}{A_1}\right)^{1/2}(Q_1-Q_0) \qquad [8]$$

Substituting [8] into [3] leads to the following expression for the pressure

$$\begin{aligned} p &= p_0+\frac{\rho c}{(A_1A_2)^{1/2}}(Q_1-Q_0) \\ &= p_0+\left(\frac{A_1}{A_2}\right)^{1/2}(p_1-p_0) \end{aligned} \qquad [9]$$

A narrowing causes an increase in the pressure and a decrease in the transmitted discharge. Note that the details of the variations of A between x_1 and x_2 does not influence the result.

Also note that $A_2 = 0$ leads to a zero discharge Q_2 and an infinite pressure p_2. This could be expected because a nonzero discharge in a pipe with a nil cross-sectional area is not physically permissible.

N.B. The reader may be interested to derive the expression of the discharge Q_3 and the pressure p_3 when the wave reflecting from $x = x_2$ reaches the abscissa x_1. Could this result be expected ?